

SUPERCONVERGENCE OF SIMPLE CONFORMING MIXED FINITE ELEMENTS FOR LINEAR ELASTICITY ON RECTANGULAR GRIDS IN ANY SPACE DIMENSION

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ABSTRACT. This paper is to prove superconvergence of a family of simple conforming mixed finite elements of first order for the linear elasticity problem with the Hellinger–Reissner variational formulation. The analysis is based on three main ingredients: a new interpolation operator, a new expansion method, and a new iterative argument for superconvergence analysis.

Keywords. mixed finite element, linear elasticity, conforming finite element, superconvergence.

AMS subject classifications. 65N30, 73C02.

1. INTRODUCTION

This paper investigates superconvergence of simple conforming mixed finite elements [27] for linear elasticity within the Hellinger-Reissner variational principle. It is well-known that it is a challenge problem for stable discretizations for this problem, which results from a strong coupling of the symmetry requirement on the discrete stress tensor and the usual stable conditions for mixed finite element methods. A lot of efforts, see, for instance, [2, 5, 6, 30, 34, 37, 38, 39], have been devoted to developing stable methods of this problem. But no stable mixed finite element was found in the first four decades [8]. Not until the year 2002, were there some advances in this direction. In [8] and [4], a sufficient condition is proposed, which states that a discrete exact sequence guarantees the stability of the mixed method. From then, conforming mixed finite elements on the simplicial and rectangular triangulations have been constructed [1, 3, 4, 8, 10, 17]; see [9, 23, 26, 33, 40, 42, 43] for nonconforming mixed finite elements, and [7, 11, 19, 24, 25] for new weakly symmetric finite elements. However, most of these elements are difficult to be implemented; numerical examples can only be found in [15, 16, 28, 27, 43] so far.

In a recent paper [27], a new family of simple, any space-dimensional, symmetric, conforming mixed finite elements for the problem is proposed. In these elements, quadratic polynomials $\{1, x_i, x_i^2\}$ are used for the normal stresses σ_{ii} , bilinear polynomials

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$\{1, x_i, x_j, x_i x_j\}$ for the shear stresses σ_{ij} , and linear polynomials $\{1, x_i\}$ for the displacements u_i . The stress and displacement spaces of [27] are actually enrichment of those in [28], a family of symmetric nonconforming elements. These are possibly the simplest conforming mixed finite element methods. A first order convergence was established for these elements in [27]. However, superconvergence was observed from numerical examples presented therein.

Superconvergence is one of the most active research fields for finite element methods. A lot of fundamental results can be found for conforming, nonconforming and mixed finite elements of model problems in literature, see for instance, [21, 31, 32]. However, no results can be found for the mixed finite element methods under consideration in literature so far. A very recent paper [36] analyzed superconvergence of a family of conforming rectangular mixed finite element methods for the two dimensional linear elasticity problem. However, in the conclusion, it was pointed out that the technique therein can not be applied to mixed elements under consideration.

The aim of this paper is to prove superconvergence observed in [27]. One challenge is that the canonical interpolation operators for the stress spaces have no commuting properties, which are indispensable ingredients for superconvergence analysis for mixed finite elements for the Poisson equations, see for instance, [12, 20, 21, 22, 41], also for the linear elasticity problem [36], and more details in Section 3.1. Another challenge is that the normal stresses are coupled and consequently the superclose analysis used in [21] for the mixed finite element of the Poisson equation can not be extended to the present case (see more details in Section 3.1.) To overcome these difficulties, we propose a new interpolation operator. Compared with the original interpolation operator from [27], the new one has a superclose property that is accomplished by adopting a new expansion which is motivated by a recent paper [29]. Finally we propose an iterative argument to establish an $O(h^{1+1/2})$ superconvergence.

This paper denotes by $H^k(T, X)$ the Sobolev space consisting of functions on domain $T \subset \mathbb{R}^n$, taking values in the finite-dimensional vector space X and having all derivatives of order at most k square-integrable. For our purposes, the range space X will be \mathbb{S} , or \mathbb{R}^n , or \mathbb{R} . In the latter case we may write simply $H^k(T)$. $\|\cdot\|_{k,T}$ is the Sobolev norm on $H^k(T)$. Here \mathbb{S} denotes the space of symmetric tensors, $H(\text{div}, T, \mathbb{S})$, consisting of square-integrable symmetric matrix fields with square-integrable divergence. The norm $\|\cdot\|_{H(\text{div}, T)}$ reads

$$\|\tau\|_{H(\text{div}, T)}^2 := \|\tau\|_{0,T}^2 + \|\text{div}\tau\|_{0,T}^2.$$

$L^2(T, \mathbb{R}^n)$ is the space of vector-valued functions which are square-integrable.

2. THE LINEAR ELASTICITY PROBLEM AND MIXED FINITE ELEMENTS

2.1. The linear elasticity problem. Based on the Hellinger-Reissner principle, the n -dimensional linear elasticity problem within a stress-displacement $(\sigma-u)$ form reads: Find $(\sigma, u) \in \Sigma \times V := H(\operatorname{div}, \Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{R}^n)$, such that

$$(2.1) \quad \begin{cases} (A\sigma, \tau) + (\operatorname{div}\tau, u) = 0 & \text{for all } \tau \in \Sigma, \\ (\operatorname{div}\sigma, v) = (f, v) & \text{for all } v \in V. \end{cases}$$

Here the symmetric tensor space for stress Σ and the space for vector displacement V are, respectively,

$$\begin{aligned} H(\operatorname{div}, \Omega, \mathbb{S}) &:= \left\{ \left(\sigma_{ij} \right)_{n \times n} \in H(\operatorname{div}, \Omega) \mid \sigma_{ij} = \sigma_{ji} \right\}, \\ L^2(\Omega, \mathbb{R}^n) &:= \left\{ \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix}^T \mid u_i \in L^2(\Omega) \right\}. \end{aligned}$$

The matrix A is defined as

$$A\sigma = \frac{1}{2\mu} \left(\sigma - \frac{\lambda}{2\mu + n\lambda} \operatorname{tr}(\sigma) \delta \right)$$

where δ is the identity matrix of $n \times n$, and μ and λ are the Lamé constants.

This paper deals with a pure displacement problem with the homogeneous boundary condition that $u \equiv 0$ on $\partial\Omega$. The domain is assumed to be a rectangular polyhedron in \mathbb{R}^n .

2.2. The n -dimensional conforming mixed finite element space. We recall a conforming mixed finite element method proposed in [27] for the problem (2.1). We shall follow the notations used therein.

The rectangular domain Ω is subdivided by a family of rectangular grids \mathcal{T}_h (with the grid size h). For convenience, the set of all $n-1$ dimensional faces in \mathcal{T}_h is denoted by \mathcal{F}_h . For all element $K \in \mathcal{T}_h$, the set of all $n-1$ dimensional faces of K perpendicular to x_i -axis is denoted by $\mathcal{F}_{x_i, K}$, the set of all $n-2$ dimensional faces of K perpendicular to x_i and x_j axes by $\mathcal{F}_{x_i, x_j, K}$. Given any face $\mathcal{F} \in \mathcal{F}_h$, one fixed unit normal vector ν with components $(\nu_1, \nu_2, \dots, \nu_n)$ is assigned.

We first introduce the finite element space locally on a single n -rectangle $K \in \mathcal{T}_h$:

$$\begin{aligned} V(K) &:= \left\{ v = (v_1, \dots, v_n) \mid v_i \in P_1(x_i) \right\}, \\ \Sigma(K) &:= \left\{ \sigma \in \left(\sigma_{ij} \right)_{\mathbb{S}} \mid \sigma_{ii} \in P_2(x_i), \sigma_{ij} \in Q_1(x_i, x_j), i \neq j \right\}, \end{aligned}$$

where

$$\begin{aligned} P_1(x_i) &:= \operatorname{span} \{1, x_i\}, \\ P_2(x_i) &:= \operatorname{span} \{1, x_i, x_i^2\}, \\ Q_1(x_i, x_j) &:= \operatorname{span} \{1, x_i, x_j, x_i x_j\}. \end{aligned}$$

For example, in 2D ($n = 2$), the spaces may be displayed as

$$\begin{aligned}\Sigma(K) &= \text{span} \begin{pmatrix} \{1, x_1, x_1^2\} & \{1, x_1, x_2, x_1 x_2\} \\ \{1, x_1, x_2, x_1 x_2\} & \{1, x_2, x_2^2\} \end{pmatrix}, \\ V(K) &= \text{span} \begin{pmatrix} \{1, x_1\} \\ \{1, x_2\} \end{pmatrix}.\end{aligned}$$

Due to the $H(\text{div})$ requirement, σ_{ii} has to be continuous in x_i direction, while σ_{ij} has to be continuous in both x_i and x_j directions. Thus, we can specify the local degrees of freedom for the two finite element spaces on element K as follows,

- $\frac{1}{|K|} \int_K u_i \, v \, dV$, for all $v \in P_1(x_i)$ and $i = 1, 2, \dots, n$;
- $\frac{1}{|F_{x_i, K}|} \int_{F_{x_i, K}} \sigma_{ii} \, dF$, for all $F_{x_i, K} \in \mathcal{F}_{x_i, K}$ and $i = 1, 2, \dots, n$;
- $\frac{1}{|K|} \int_K \sigma_{ii} \, dV$, $i = 1, 2, \dots, n$;
- $\frac{1}{|F_{x_i, x_j, K}|} \int_{F_{x_i, x_j, K}} \sigma_{ij} \, dF$, for all $F_{x_i, x_j, K} \in \mathcal{F}_{x_i, x_j, K}$ and $1 \leq i < j \leq n$.

The global spaces Σ_h and V_h can be defined by their property

$$(2.2) \quad \Sigma_h := \{ \sigma \in H(\text{div}, \Omega, \mathbb{S}) \mid \sigma|_K \in \Sigma(K) \text{ for all } K \in \mathcal{T}_h \},$$

$$(2.3) \quad V_h := \{ v \in L^2(\Omega, \mathbb{R}^n) \mid v|_K \in V(K) \text{ for all } K \in \mathcal{T}_h \}.$$

The mixed finite element approximation of Problem (2.1) reads: Find $(\sigma_h, u_h) \in \Sigma_h \times V_h$ such that

$$(2.4) \quad \begin{cases} (A\sigma_h, \tau) + (\text{div} \tau, u_h) = 0 & \text{for all } \tau \in \Sigma_h, \\ (\text{div} \sigma_h, v) = (f, v) & \text{for all } v \in V_h. \end{cases}$$

It follows from the definition of V_h and Σ_h that

$$\text{div } \Sigma_h \subset V_h.$$

This, in turn, leads to a strong divergence-free space:

$$(2.5) \quad \begin{aligned} Z_h &:= \{ \tau_h \in \Sigma_h \mid (\text{div } \tau_h, v) = 0 \text{ for all } v \in V_h \} \\ &= \{ \tau_h \in \Sigma_h \mid \text{div } \tau_h = 0 \text{ pointwise} \}. \end{aligned}$$

2.3. Well-posedness of the discrete problem. The well-posedness of the discrete problem (2.4) is proved in [27]. More precisely, it was shown therein that:

- (1) K-ellipticity. There exists a constant $C > 0$, independent of the meshsize h , such that

$$(2.6) \quad (A\tau, \tau) \geq C \|\tau\|_{H(\text{div})}^2 \text{ for all } \tau \in Z_h,$$

where Z_h is the divergence-free space defined in (2.5).

- (2) Discrete B-B condition. There exists a positive constant $C > 0$ independent of the meshsize h , such that

$$(2.7) \quad \inf_{0 \neq v \in V_h} \sup_{0 \neq \tau \in \Sigma_h} \frac{(\operatorname{div} \tau, v)}{\|\tau\|_{H(\operatorname{div})} \|v\|_0} \geq C.$$

In addition, there is a refined discrete inf-sup condition from [27] as follows:

Lemma 2.1. *For any $v \in V_h$, there exists an $H(\operatorname{div})$ field*

$$\tau = \begin{pmatrix} \tau_{11} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \tau_{ii} & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \tau_{nn} \end{pmatrix} \in \Sigma_h,$$

such that

$$(2.8) \quad \operatorname{div} \tau = v \text{ and } \frac{(\operatorname{div} \tau, v)}{\|\tau\|_{H(\operatorname{div})}} \geq \sqrt{\frac{2}{3}} \|v\|_0.$$

2.4. Error estimate. Since it is very difficult to show the superclose property of the interpolation operator defined in [27], for any $\sigma \in H^2(\Omega, \mathbb{S})$, we define a new interpolation by

$$\Pi_h \sigma = \begin{pmatrix} \Pi_{11} \sigma_{11} & \Pi_{12} \sigma_{12} & \cdots & \cdots & \Pi_{1n} \sigma_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \Pi_{i1} \sigma_{i1} & \cdots & \Pi_{ii} \sigma_{ii} & \cdots & \Pi_{in} \sigma_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Pi_{n1} \sigma_{n1} & \Pi_{n2} \sigma_{n2} & \cdots & \cdots & \Pi_{nn} \sigma_{nn} \end{pmatrix} \in \Sigma_h,$$

where $\Pi_{ij} = \Pi_{ji}$, are defined next. The interpolation operator Π_{ii} is defined by, for any $K \in \mathcal{T}_h$,

$$\begin{aligned} \int_{F_{x_i, K}} \Pi_{ii} \sigma_{ii} dF &= \int_{F_{x_i, K}} \sigma_{ii} dF && \text{for all } F_{x_i, K} \in \mathcal{F}_{x_i, K}, \\ \int_K \Pi_{ii} \sigma_{ii} dV &= \int_K \sigma_{ii} dV && \text{for all } K \in \mathcal{T}_h. \end{aligned}$$

Here and throughout this paper, $\mathcal{F}_{x_i, K}$ denotes the set of $n-1$ dimensional faces of K which are perpendicular to the x_i axis and $\mathcal{F}_{x_i} = \cup_{K \in \mathcal{T}_h} \mathcal{F}_{x_i, K}$.

To define Π_{ij} , we introduce the nodal basis functions on unit element.

$$\phi_k(x, y) := \begin{cases} (x-1)(y-1), & k=0, \\ -(x-0)(y-1), & k=1, \\ (x-0)(y-0), & k=2, \\ -(x-1)(y-0), & k=3. \end{cases}$$

We also use multi-index notations as follows

$$(2.9) \quad \sum_{L_{\mathbf{i}, \mathbf{j}}=1}^N := \sum_{\{1 \leq l_k \leq N | k \neq i, k \neq j\}} \dots \sum.$$

Since $\sigma_{ij} \in H^2(\Omega)$, these basis functions and the short notation allow for defining the interpolation as follows

$$(2.10) \quad \Pi_{ij}\sigma_{ij} = \sum_{L_{\mathbf{i}, \mathbf{j}}=1}^N \sum_{l_i, l_j=1}^N \sum_{k=0}^3 c_{l_1, \dots, l_n}^{(ij), k} \phi_k\left(\frac{x_i}{h} - (l_i - 1), \frac{x_j}{h} - (l_j - 1)\right)$$

where the interpolation parameters satisfy

$$\begin{aligned} & c_{l_1, \dots, (l_i-1), \dots, (l_j-1), \dots, l_n}^{(ij), 2} \\ &= c_{l_1, \dots, l_i, \dots, (l_j-1), \dots, l_n}^{(ij), 3} = c_{l_1, \dots, l_i, \dots, l_j, \dots, l_n}^{(ij), 0} = c_{l_1, \dots, (l_i-1), \dots, l_j, \dots, l_n}^{(ij), 1} \\ &= \frac{1}{|F_{x_i, x_j, l_1, \dots, l_i, \dots, l_j, \dots, l_n}|} \int_{F_{x_i, x_j, l_1, \dots, l_i, \dots, l_j, \dots, l_n}} \sigma_{ij} dF, \quad 0 < l_i, l_j < N, \end{aligned}$$

where $F_{x_i, x_j, l_1, \dots, l_i, \dots, l_j, \dots, l_n}$ is the unique $n-2$ dimensional face at vertex

$$((l_1 - 1)h, \dots, (l_i - 1)h, \dots, (l_j - 1)h, \dots, (l_n - 1)h)$$

which is shared by elements:

$$\begin{aligned} K_1 &= [l_1, \dots, l_i, \dots, l_j, \dots, l_n], \\ K_2 &= [l_1, \dots, (l_i - 1), \dots, (l_j - 1), \dots, l_n], \\ K_3 &= [l_1, \dots, l_i, \dots, (l_j - 1), \dots, l_n], \\ K_4 &= [l_1, \dots, (l_i - 1), \dots, l_j, \dots, l_n], \end{aligned}$$

$|F_{x_i, x_j, l_1, \dots, l_i, \dots, l_j, \dots, l_n}|$ is the measure of face $F_{x_i, x_j, l_1, \dots, l_i, \dots, l_j, \dots, l_n}$. If

$$\begin{aligned} & |F_{x_i, x_j, l_1, \dots, l_i, \dots, l_j, \dots, l_n}| = 0, \\ & \frac{1}{|F_{x_i, x_j, l_1, \dots, l_i, \dots, l_j, \dots, l_n}|} \int_{F_{x_i, x_j, l_1, \dots, l_i, \dots, l_j, \dots, l_n}} \sigma_{ij} dF \end{aligned}$$

is understood as the value of σ_{ij} at vertex

$$((l_1 - 1)h, \dots, (l_i - 1)h, \dots, (l_j - 1)h, \dots, (l_n - 1)h).$$

The operator Π_{ij} is different from that defined in [27]. Note that the corresponding operator of [27] can not be used for the superconvergence analysis. For this operator, we have the following error estimates:

$$(2.11) \quad \|\sigma_{ij} - \Pi_{ij}\sigma_{ij}\|_{0,K} \leq Ch\|\sigma_{ij}\|_{2,K},$$

$$(2.12) \quad \left\| \frac{\partial}{\partial x_i}(\sigma_{ij} - \Pi_{ij}\sigma_{ij}) \right\|_{0,K} \leq Ch\|\sigma_{ij}\|_{2,K},$$

$$(2.13) \quad \left\| \frac{\partial}{\partial x_j}(\sigma_{ij} - \Pi_{ij}\sigma_{ij}) \right\|_{0,K} \leq Ch\|\sigma_{ij}\|_{2,K}.$$

Since the space for the operator Π_{ii} contains the 1D quadratic polynomials $\text{span}\{1, x_i, x_i^2\}$, the scaling argument and standard approximation, state

$$(2.14) \quad |\sigma_{ii} - \Pi_{ii}\sigma_{ii}|_{0,K} \leq Ch|\sigma_{ii}|_{1,K},$$

$$(2.15) \quad \left| \frac{\partial}{\partial x_i}(\sigma_{ii} - \Pi_{ii}\sigma_{ii}) \right|_{0,K} \leq Ch \left| \frac{\partial \sigma_{ii}}{\partial x_i} \right|_{1,K},$$

for all $K \in \mathcal{T}_h$.

A summary of these aforementioned estimates (2.11)–(2.15) leads to

Theorem 2.1. *For any $\sigma \in H^2(\Omega, \mathbb{S})$, we have that*

$$(2.16) \quad \|\sigma - \Pi_h\sigma\|_0 \leq Ch\|\sigma\|_2,$$

$$(2.17) \quad \|\text{div}(\sigma - \Pi_h\sigma)\|_0 \leq Ch\|\sigma\|_2.$$

The stability of the elements and the standard theory of mixed finite element methods, see for instance [13, 14], give the following abstract error estimate:

$$(2.18) \quad \|\sigma - \sigma_h\|_{H(\text{div})} + \|u - u_h\|_0 \leq C \inf_{\tau_h \in \Sigma_h, v_h \in V_h} (\|\sigma - \tau_h\|_{H(\text{div})} + \|u - v_h\|_0).$$

Let P_h denotes the projection operator from V to V_h , which has the error estimate

$$(2.19) \quad \|v - P_h v\|_0 \leq Ch\|v\|_1.$$

Choosing $\tau_h = \Pi_h\sigma$ and $v_h = P_h u$ in (2.18), the estimates (2.16), (2.17), (2.19) prove

Theorem 2.2. *Let $(\sigma, u) \in \Sigma \times V$ be the exact solution of problem (2.1) and $(\tau_h, u_h) \in \Sigma_h \times V_h$ the finite element solution of (2.4). Then,*

$$\|\sigma - \sigma_h\|_{H(\text{div})} \leq Ch(\|\sigma\|_2 + \|u\|_1),$$

$$\|u - u_h\|_0 \leq Ch(\|\sigma\|_2 + \|u\|_1).$$

3. THE SUPERCLOSE PROPERTY OF THE CANONICAL INTERPOLATIONS

3.1. Main difficulties. By the K-ellipticity in (2.6) and the discrete inf-sup condition (2.7), it is routine to prove that

$$(3.1) \quad \begin{aligned} & \|\Pi_h \sigma - \sigma_h\|_{H(\text{div})} + \|P_h u - u_h\|_0 \\ & \leq C \sup_{0 \neq (\tau, v) \in \Sigma_h \times V_h} \frac{(A(\sigma - \Pi_h \sigma), \tau) + (u - P_h u, \text{div } \tau) - (\text{div}(\sigma - \Pi_h \sigma), v)}{\|\tau\|_{H(\text{div})} + \|v\|_0}. \end{aligned}$$

Note that the inequality (3.1) is the starting point for superconvergence analysis of mixed finite element methods, see for instance, [12, 21] and [36]. However, this formulation can not be directly used for mixed finite elements under consideration, the reasons lie in that

- The interpolation operator lacks the usual commuting property, namely,

$$\text{div } \Pi_h \sigma \neq P_h \text{div } \sigma.$$

- The components of the stress normal are coupled through $A(\sigma - \Pi_h \sigma)$, it is impossible to prove directly the following super-close property

$$(A(\sigma - \Pi_h \sigma), \tau) \leq Ch^2 \|\sigma\|_2 \|\tau\|_{H(\text{div})}$$

for a general $\tau \in \Sigma_h$.

In the sequel, we will need some results on Sobolev spaces. They are formulated in the following lemma. First of all, define $\partial\Omega_h$ as the subset of points having (Euclidian) distance less than h from the boundary:

$$\partial\Omega_h := \{x \in \Omega \mid \exists y \in \partial\Omega : \text{dist}(x, y) \leq h\}.$$

Lemma 3.1. *For $v \in H^s(\Omega)$ with $0 \leq s \leq 1/2$, it holds*

$$(3.2) \quad \|v\|_{0, \partial\Omega_h} \leq Ch^s \|v\|_s.$$

3.2. The superclose property of $(\text{div}(\sigma - \Pi_h \sigma), v)$. To overcome the first difficulty, we follow the idea of [29] to adopt a new expansion of the operator Π_h . In fact, let $\Pi_K = \Pi_h|_K$, we have the following crucial result.

Lemma 3.2. *For any $\sigma \in P_2(K, \mathbb{S})$ and $v \in V_K$, it holds that*

$$(3.3) \quad (\text{div}(\sigma - \Pi_K \sigma), v)_K = 0.$$

Proof. We only need to prove the result on the reference element $K = [-1, 1]^n$. For any $\sigma \in P_2(K, \mathbb{S})$, its components σ_{ij} , $i, j = 1, \dots, n$ with $i \neq j$, can be expressed as

$$\sigma_{ij} = p_0(x_i, x_j) + x_i p_1^{(ij)} + x_j p_2^{(ij)} + p_3^{(ij)},$$

where $p_0(x_i, x_j)$ is a polynomial of degree 2 in x_i and x_j , both $p_1^{(ij)}$ and $p_2^{(ij)}$ are homogeneous polynomials of degree 1 of $(n-2)$ of variables x_k , $k = 1, \dots, n$ with

$k \neq i, j$, and $p_3^{(ij)} \in P_2(K)$ is a polynomial of degree 2 with respect to $(n-2)$ variables x_k , $k = 1, \dots, n$ with $k \neq i, j$. The definition of $\Pi_{ij,K}$ leads to

$$(3.4) \quad \begin{aligned} \sigma_{ij} - \Pi_{ij,K} \sigma_{ij} &= c_{ii}(x_i^2 - 1) + c_{ij}(x_j^2 - 1) + x_i p_1^{(ij)} + x_j p_2^{(ij)} \\ &+ p_3^{(ij)} - \frac{1}{|F_{x_i, x_j}|} \int_{F_{x_i, x_j}} p_3^{(ij)} dF, \end{aligned}$$

for two interpolation parameters c_{ii} and c_{ij} , where F_{x_i, x_j} is any $n-2$ dimensional face of K which is perpendicular to the plane $\text{span}\{x_i, x_j\}$. Here we use the facts that $\int_{F_{x_i, x_j}} p_1^{(ij)} dF = \int_{F_{x_i, x_j}} p_2^{(ij)} dF = 0$, and that $p_3^{(ij)}$ is a constant function with respect to variables x_i and x_j , and that $\sum_{k=0}^3 \phi_k = 1$.

If $i = j$, we have

$$\sigma_{ii} = p_0(x_i) + x_i p_1^{(ii)} + p_2^{(ii)},$$

where $p_0(x_i)$ is a polynomial of degree 2 or less in one variable x_i , and $p_1^{(ii)}$ is a homogeneous polynomial of degree 1 of $(n-1)$ variables x_k , $k = 1, \dots, n$ with $k \neq i$, and $p_2^{(ii)} \in P_2(K)$ is a polynomial of degree 2 of $(n-1)$ variables x_k , $k = 1, \dots, n$ with $k \neq i$. The definition of $\Pi_{ii,K}$ leads to

$$(3.5) \quad \sigma_{ii} - \Pi_{ii,K} \sigma_{ii} = x_i p_1^{(ii)} + p_2^{(ii)} - \frac{1}{|K|} \int_K p_2^{(ii)} dV.$$

By (3.4) and (3.5), we have

$$\frac{\partial(\sigma_{ij} - \Pi_{ij,K} \sigma_{ij})}{\partial x_j} = p_2^{(ij)} + 2c_{ij}x_j \quad \text{and} \quad \frac{\partial(\sigma_{ii} - \Pi_{ii,K} \sigma_{ii})}{\partial x_i} = p_1^{(ii)}.$$

Hence, the i -th component of $\text{div}(\sigma - \Pi_K \sigma)$ can be expressed as

$$(\text{div}(\sigma - \Pi_K \sigma))_i = p_1^{(ii)} + \sum_{i \neq j=1}^n (p_2^{(ij)} + 2c_{ij}x_j).$$

We can compute

$$\begin{aligned} \int_K (\text{div}(\sigma - \Pi_K \sigma))_i dV &= \int_K p_1^{(ii)} dV + \sum_{i \neq j=1}^n \int_K (p_2^{(ij)} + 2c_{ij}x_j) dV \\ &= 0 + \sum_{i \neq j=1}^n \left(\int_K p_2^{(ij)} dV + 2c_{ij} \int_K x_j dV \right) = 0, \\ \int_K (\text{div}(\sigma - \Pi_K \sigma))_i x_i dV &= \int_{-1}^1 x_i dx_i \int_{K_{n-1}} \left(p_1^{(ii)} + \sum_{i \neq j=1}^n (p_2^{(ij)} + 2c_{ij}x_j) \right) dV_{n-1} \\ &= 0 \cdot 0 = 0, \end{aligned}$$

Here $K_{n-1} = [0, 1]^{n-1}$ is the $n-1$ dimensional cube without variable x_i . Note that the i -th component of v can be written as

$$v_i = a_0 + a_1 x_i$$

for two interpolation parameters a_0 and a_1 . Thus

$$((\operatorname{div}(\sigma - \Pi_K \sigma))_i, v_i)_K = 0,$$

which completes the proof. \blacksquare

As a consequence of (3.3), we have the following superclose property for the term $(\operatorname{div}(\sigma - \Pi_h \sigma), v)$.

Lemma 3.3. *Suppose that $\sigma \in H^3(\Omega, \mathbb{S})$. Then it holds that*

$$(3.6) \quad |(\operatorname{div}(\sigma - \Pi_h \sigma), v)| \leq Ch^2 |\sigma|_3 \|v\|_0 \text{ for any } v \in V_h.$$

Proof. Given element K , let $I_{2,K} : H(\operatorname{div}, K, \mathbb{S}) \rightarrow P_2(K, \mathbb{S})$ be the L^2 projection operator defined as: Given $\tau \in H(\operatorname{div}, K, \mathbb{S})$, find $I_{2,K}\tau \in P_2(K, \mathbb{S})$ such that

$$\int_K I_{2,K}\tau q dV = \int_K \tau q dV \text{ for any } q \in P_2(K, \mathbb{S}).$$

This allows for the following decomposition:

$$\begin{aligned} (\operatorname{div}(\sigma - \Pi_h \sigma), v) &= \sum_{K \in \mathcal{T}_h} (\operatorname{div}((I - \Pi_K)I_{2,K}\sigma + (I - \Pi_K)(I - I_{2,K})\sigma), v)_K \\ &= \sum_{K \in \mathcal{T}_h} (\operatorname{div}((I - \Pi_K)(I - I_{2,K})\sigma), v)_K, \end{aligned}$$

where we applied (3.3). The desired result follows from the stability of Π_h and the approximation property of $I_{2,K}$. \blacksquare

3.3. The superclose property of $(A(\sigma - \Pi_h \sigma), \sigma_h - \Pi_h \sigma)$. To deal with the second difficulty, we propose to explore the strong discrete inf-sup condition presented in Lemma 2.1.

Lemma 3.4. *For any $\sigma \in P_1(K, \mathbb{S})$ and $\tau \in \Sigma_{n,h}$, it holds that*

$$(3.7) \quad |(A(\sigma - \Pi_K \sigma), \tau)_K| \leq Ch^2 \|\operatorname{div} \tau\|_{0,K} |\sigma|_{1,K},$$

where

$$\Sigma_{n,h} = \{\tau = \operatorname{diag}(\tau_{11}, \dots, \tau_{nn}), \tau \in \Sigma_h\}.$$

Proof. We only need to prove the result on the reference element $K = [-1, 1]^n$. For any $\sigma \in P_1(K, \mathbb{S})$, its normal components can be written as

$$\sigma_{ii} = c_0^{(ii)} + \sum_{j=1}^n c_j^{(ii)} x_j, \quad i = 1, \dots, n,$$

where $c_j^{(ii)}$, $j = 0, \dots, n$, are interpolation parameters. By the definition of the operator A , the ii -th component of $A(\sigma - \Pi_K \sigma)$ is

$$A(\sigma - \Pi_K \sigma)_{ii} = \frac{1}{2\mu(2\mu + n\lambda)} \left((2\mu + (n-1)\lambda) \sum_{i \neq j=1}^n c_j^{(ii)} x_j - \lambda \sum_{i \neq k=1}^n \sum_{k \neq j=1}^n c_j^{(kk)} x_j \right).$$

Note that the i -th component of τ can be written as

$$\tau_{ii} = a_0^{(ii)} + a_1^{(ii)} x_i + a_2^{(ii)} x_i^2$$

for parameters $a_0^{(ii)}$, $a_1^{(ii)}$ and $a_2^{(ii)}$. Therefore,

$$\begin{aligned} (A(\sigma - \Pi_K \sigma)_{ii}, \tau_{ii})_K &= -a_1^{(ii)} \frac{\lambda}{2\mu(2\mu + n\lambda)} \sum_{i \neq k=1}^n c_i^{(kk)}(x_i, x_i) \\ (3.8) \quad &= -\frac{2^n}{3} \frac{\partial \tau_{ii}}{\partial x_i}(0) \frac{\lambda}{2\mu(2\mu + n\lambda)} \sum_{i \neq k=1}^n \frac{\partial \sigma_{kk}}{\partial x_i}, \end{aligned}$$

$$|(A(\sigma - \Pi_K \sigma)_{ii}, \tau_{ii})_K| \leq C \frac{\lambda}{6\mu(2\mu + n\lambda)} \left\| \frac{\partial \tau_{ii}}{\partial x_i} \right\|_{0,K} \left\| \sum_{i \neq k=1}^n \frac{\partial \sigma_{kk}}{\partial x_i} \right\|_{0,K}.$$

A summation over all n components leads to

$$|(A(\sigma - \Pi_K \sigma), \tau)_K| \leq C \|\operatorname{div} \tau\|_{0,K} |\sigma|_{1,K}.$$

The final result follows from a scaling argument. ■

A combination of the above lemma and (3.6) yields the following important result.

Lemma 3.5. *It holds that*

$$(3.9) \quad \|\Pi_h \sigma - \sigma_h\|_0^2 + \|P_h u - u_h\|_0^2 \leq C(A(\Pi_h \sigma - \sigma), \Pi_h \sigma - \sigma_h) + Ch^4 \|\sigma\|_3^2.$$

Proof. It follows from Lemma 2.1 that there exists $\tau = \operatorname{diag}(\tau_{11}, \dots, \tau_{nn}) \in \Sigma_h$ such that

$$(3.10) \quad \operatorname{div} \tau = u_h - P_h u \text{ and } \|\tau\|_{H(\operatorname{div})} \leq C \|u_h - P_h u\|_0.$$

This allows for the following decomposition:

$$\begin{aligned} (u_h - P_h u, u_h - P_h u) &= (u_h - P_h u, \operatorname{div} \tau) = (u_h - u, \operatorname{div} \tau) = (A(\sigma - \sigma_h), \tau) \\ &= (A(\sigma - \Pi_h \sigma), \tau) + (A(\Pi_h \sigma - \sigma_h), \tau). \end{aligned}$$

Since τ is a diagonal matrix, it follows from (3.7) and (3.10) that

$$(A(\sigma - \Pi_h \sigma), \tau) \leq Ch^2 |\sigma|_1 \|\operatorname{div} \tau\|_0 \leq Ch^2 |\sigma|_1 \|u_h - P_h u\|_0.$$

A substitution of this inequality into the previous equation, by the Cauchy–Schwarz inequality and (3.10), leads to

$$(3.11) \quad \|u_h - P_h u\|_0 \leq C(h^2 |\sigma|_1 + \|\Pi_h \sigma - \sigma_h\|_0).$$

On the other hand, let $\tau = \Pi_h \sigma - \sigma_h$, we have

$$\begin{aligned}
 (3.12) \quad (A(\Pi_h \sigma - \sigma_h), \tau) &= (A(\Pi_h \sigma - \sigma), \tau) + (A(\sigma - \sigma_h), \tau) \\
 &= (A(\Pi_h \sigma - \sigma), \tau) - (u - u_h, \operatorname{div} \tau) \\
 &= (A(\Pi_h \sigma - \sigma), \tau) - (P_h u - u_h, \operatorname{div}(\Pi_h \sigma - \sigma_h)) \\
 &= (A(\Pi_h \sigma - \sigma), \tau) - (P_h u - u_h, \operatorname{div}(\Pi_h \sigma - \sigma)).
 \end{aligned}$$

From (3.6) and (3.11) it follows

$$\begin{aligned}
 (3.13) \quad (A(\Pi_h \sigma - \sigma_h), \tau) &\leq (A(\Pi_h \sigma - \sigma), \tau) + Ch^2 |\sigma|_3 \|P_h u - u_h\|_0 \\
 &\leq (A(\Pi_h \sigma - \sigma), \tau) + Ch^2 |\sigma|_3 (h^2 |\sigma|_1 + \|\Pi_h \sigma - \sigma_h\|_0).
 \end{aligned}$$

Since there exists a positive constant β such that

$$\beta \|\Pi_h \sigma - \sigma_h\|_0^2 \leq (A(\Pi_h \sigma - \sigma_h), \tau),$$

an application of the Young inequality leads to

$$\|\Pi_h \sigma - \sigma_h\|_0^2 + \|P_h u - u_h\|_0^2 \leq C(A(\Pi_h \sigma - \sigma), \tau) + Ch^4 \|\sigma\|_3^2,$$

which completes the proof. ■

Lemma 3.6. *For any $\sigma_{ij} \in P_1(K)$ and $\tau_{ij} \in Q_1(x_i, x_j)$, it holds that*

$$(3.14) \quad (\sigma_{ij} - \Pi_{ij,K} \sigma_{ij}, \tau_{ij})_K = 0.$$

Proof. We only need to prove the result on the reference element $K = [-1, 1]^n$. Since $\sigma_{ij} \in P_1(K)$, we have

$$\sigma_{ij} - \Pi_{ij,K} \sigma_{ij} = p^{(ij)},$$

where $p^{(ij)}$ be a homogeneous polynomial of degree 1 with respect to variables x_k , $k = 1, \dots, n$ but $k \neq i, j$. Any $\tau_{ij} \in Q_1(x_i, x_j)$ can be expressed as

$$\tau_{ij} = a_0 + a_1 x_i + a_2 x_j + a_3 x_i x_j,$$

for four interpolation parameters a_k , $k = 0, \dots, 3$. On the reference element K , it is straightforward to see that

$$(\sigma_{ij} - \Pi_{ij,K} \sigma_{ij}, \tau_{ij}) = 0.$$

This completes the proof. ■

This lemma and a similar argument of (3.6) can prove the following supercloseness.

Lemma 3.7. *For any $\tau_{ij} \in \Sigma_{ij,h} := \{e'_i \tau e_j, \tau \in \Sigma_h\}$ it holds that*

$$(3.15) \quad (\sigma_{ij} - \Pi_{ij} \sigma_{ij}, \tau_{ij}) \leq Ch^2 |\sigma_{ij}|_2 \|\tau_{ij}\|_0,$$

provided that $\sigma_{ij} \in H^2(\Omega)$. Here e_i and e_j are the i -th and j -th canonical basis of the space \mathbb{R}^n , respectively.

Lemma 3.8. *Let (σ, u) and (σ_h, u_h) be solutions of problems (2.1) and (2.4), respectively. Suppose that $\sigma \in H^2(\Omega, \mathbb{S})$ and $u \in H^1(\Omega, \mathbb{R}^n)$. Then there holds that*

$$(3.16) \quad (A(\sigma - \Pi_h \sigma), \sigma_h - \Pi_h \sigma) \leq Ch^{5/2}(\|\sigma\|_2 + \|u\|_1)\|\sigma\|_2.$$

Proof. Let $\tau = \sigma_h - \Pi_h \sigma$. Given element K , let $I_{1,K} : L^2(K) \rightarrow P_1(K)$ be the L^2 projection operator defined as: Given $v \in L^2(K)$, find $I_{1,K}v \in P_1(K)$ such that

$$\int_K I_{1,K} v q dV = \int_K v q dV \text{ for any } q \in P_1(K, \mathbb{S}).$$

This leads to the following decomposition:

$$\begin{aligned} (A(\sigma - \Pi_h \sigma)_{ii}, \tau_{ii}) &= \sum_{K \in \mathcal{T}_h} (A(\sigma - \Pi_K \sigma)_{ii}, \tau_{ii})_K \\ &= \sum_{K \in \mathcal{T}_h} (A((I - \Pi_K)I_{1,K}\sigma)_{ii}, \tau_{ii})_K + \sum_{K \in \mathcal{T}_h} (A((I - \Pi_K)(I - I_{1,K})\sigma)_{ii}, \tau_{ii})_K. \end{aligned}$$

Then it follows from (3.8) that

$$(A(\sigma - \Pi_h \sigma)_{ii}, \tau_{ii}) = -\frac{\lambda h^2}{24\mu(2\mu + n\lambda)} \sum_{i \neq k=1}^n \sum_{K \in \mathcal{T}_h} \left(\frac{\partial(I_{1,K}\sigma)_{kk}}{\partial x_i}, \frac{\partial \tau_{ii}}{\partial x_i} \right)_K + Ch^2 \|\sigma\|_2 \|\tau_{ii}\|_0.$$

Since $\frac{\partial \tau_{ii}}{\partial x_i}$ is of the form $a_1^{(ii)} + a_2^{(ii)} x_i$ for parameters $a_1^{(ii)}$ and $a_2^{(ii)}$, $\left(\frac{\partial(I_{1,K}x_l^2 - x_l^2)}{\partial x_i}, \frac{\partial \tau_{ii}}{\partial x_i} \right)_K = 0$ for $l \neq i$. Therefore,

$$\left(\frac{\partial(I_{1,K}\sigma - \sigma)_{kk}}{\partial x_i}, \frac{\partial \tau_{ii}}{\partial x_i} \right)_K = -\frac{h}{3} \left(\frac{\partial^2 \sigma_{kk}}{\partial x_i^2}, \frac{\partial \tau_{ii}}{\partial x_i} \right)_K + Ch^2 |\sigma|_{3,K} |\tau_{ii}|_{1,K}.$$

After an elementwise inverse estimate, a combination of these two equations yield

$$(3.17) \quad (A(\sigma - \Pi_h \sigma)_{ii}, \tau_{ii}) = -\frac{\lambda h^2}{24\mu(2\mu + n\lambda)} \sum_{i \neq k=1}^n \left(\frac{\partial \sigma_{kk}}{\partial x_i}, \frac{\partial \tau_{ii}}{\partial x_i} \right) + Ch^2 \|\sigma\|_2 \|\tau_{ii}\|_0.$$

Since convergence of terms $\frac{\partial \tau_{ii}}{\partial x_i}$ is unclear, we can not obtain directly supercloseness from the previous equation. The remedy is to use convergence of divergence $\sum_{\ell=1}^n \frac{\partial \tau_{i\ell}}{\partial x_\ell}$ and continuity of $\tau_{i\ell}$ across $n-1$ dimensional interior faces which are perpendicular to the axis x_ℓ . This idea leads to the following decomposition:

$$(3.18) \quad \sum_{i \neq k=1}^n \left(\frac{\partial \sigma_{kk}}{\partial x_i}, \frac{\partial \tau_{ii}}{\partial x_i} \right) = \sum_{\ell=1}^n \sum_{i \neq k=1}^n \left(\frac{\partial \sigma_{kk}}{\partial x_i}, \frac{\partial \tau_{i\ell}}{\partial x_\ell} \right) - \sum_{i \neq \ell=1}^n \sum_{i \neq k=1}^n \left(\frac{\partial \sigma_{kk}}{\partial x_i}, \frac{\partial \tau_{i\ell}}{\partial x_\ell} \right).$$

Since $\tau_{i\ell} = \sigma_{i\ell,h} - \Pi_{i\ell} \sigma_{i\ell}$, the first term on the right-hand side of (3.18) can be estimated by the error estimates presented in (2.17) for the interpolation operator Π_h and convergence from Theorem 2.2 for the finite element solution σ_h . This yields

$$(3.19) \quad \sum_{\ell=1}^n \sum_{i \neq k=1}^n \left(\frac{\partial \sigma_{kk}}{\partial x_i}, \frac{\partial \tau_{i\ell}}{\partial x_\ell} \right) \leq Ch(\|\sigma\|_2^2 + \|u\|_1^2).$$

To analyze the second term on the right-hand side of (3.18), we shall explore the continuity to transfer integrations on the volume to integrations on the boundary and use Lemma 3.1. In fact, since the jump $[\tau_{i\ell}]_F$ across face F vanishes for interior face $F \in \mathcal{F}_{x_\ell}$, we have

$$\begin{aligned} \left(\frac{\partial \sigma_{kk}}{\partial x_i}, \frac{\partial \tau_{i\ell}}{\partial x_\ell} \right) &= - \left(\frac{\partial^2 \sigma_{kk}}{\partial x_i \partial x_\ell}, \tau_{i\ell} \right) + \sum_{F \in \mathcal{F}_{x_\ell}} \int_F [\tau_{i\ell}]_F \frac{\partial \sigma_{kk}}{\partial x_i} dF \\ (3.20) \quad &= - \left(\frac{\partial^2 \sigma_{kk}}{\partial x_i \partial x_\ell}, \tau_{i\ell} \right) + \sum_{F \in \mathcal{F}_{x_\ell} \cap \partial\Omega} \int_F \tau_{i\ell} \frac{\partial \sigma_{kk}}{\partial x_i} dF. \end{aligned}$$

In order to use Lemma 3.1, for any $F \in \mathcal{F}_{x_\ell} \cap \partial\Omega$, let K_F be the unique element such that F is one of its $n - 1$ dimensional faces. Given $v \in L^2(K_F)$, define the constant projection $\Pi_{K_F}^0 v$ by

$$\Pi_{K_F}^0 v := \frac{1}{|K_F|} \int_{K_F} v dV.$$

This, the trace theorem, inverse estimate and triangle inequality lead to

$$\begin{aligned} \int_F \tau_{i\ell} \frac{\partial \sigma_{kk}}{\partial x_i} dF &= \int_F \tau_{i\ell} (I - \Pi_{K_F}^0) \frac{\partial \sigma_{kk}}{\partial x_i} dF + \Pi_{K_F}^0 \frac{\partial \sigma_{kk}}{\partial x_i} \int_F \tau_{i\ell} dF \\ (3.21) \quad &\leq C \|\tau_{i\ell}\|_{0,K_F} |\sigma_{kk}|_{2,K_F} + Ch^{-1} \|\Pi_{K_F}^0 \frac{\partial \sigma_{kk}}{\partial x_i}\|_{0,K_F} \|\tau_{i\ell}\|_{0,K_F} \\ &\leq C \|\tau_{i\ell}\|_{0,K_F} |\sigma_{kk}|_{2,K_F} + Ch^{-1} \left\| \frac{\partial \sigma_{kk}}{\partial x_i} \right\|_{0,K_F} \|\tau_{i\ell}\|_{0,K_F}. \end{aligned}$$

Summing over F in $\mathcal{F}_{x_\ell} \cap \partial\Omega$ and taking in account the error estimates presented in (2.17) for the interpolation operator Π_h and convergence from Theorem 2.2 for the finite element solution σ_h , we arrive at

$$\sum_{F \in \mathcal{F}_{x_\ell} \cap \partial\Omega} \int_F \tau_{i\ell} \frac{\partial \sigma_{kk}}{\partial x_i} dF \leq C(\|\sigma\|_2 + \|u\|_1)(h|\sigma|_{2,\partial\Omega_h} + |\sigma|_{1,\partial\Omega_h}).$$

Hence it follows from Lemma 3.1 that

$$(3.22) \quad \sum_{F \in \mathcal{F}_{x_\ell} \cap \partial\Omega} \int_F \tau_{i\ell} \frac{\partial \sigma_{kk}}{\partial x_i} dF \leq C(\|\sigma\|_2 + \|u\|_1)(h|\sigma|_{2,\Omega} + h^{1/2}\|\sigma\|_{3/2,\Omega}).$$

A summary of (3.17) through (3.22) shows that

$$(3.23) \quad \sum_{i=1}^n (A(\sigma - \Pi_h \sigma)_{ii}, \tau_{ii}) \leq Ch^{5/2}(\|\sigma\|_2 + \|u\|_1)\|\sigma\|_2.$$

Finally, let $\sigma_n = \text{diag}(\sigma_{11}, \dots, \sigma_{nn})$, $\tau_n = \text{diag}(\tau_{11}, \dots, \tau_{nn})$, $\sigma_s = \sigma - \sigma_n$ and $\tau_s = \tau - \tau_n$. The previous equation, the estimate (3.15) for the shear stress, estimates (2.16)–(2.17), and estimates in Theorem 2.2, yield

$$(A(\sigma - \Pi_h \sigma), \tau) = (A(\sigma_n - \Pi_h \sigma_n), \tau_n) + (\sigma_s - \Pi_h \sigma_s, \tau_s) \leq Ch^{5/2}(\|\sigma\|_2 + \|u\|_1)\|\sigma\|_2.$$

This completes the proof. ■

Theorem 3.1. *Let (σ, u) and (σ_h, u_h) be solutions of problems (2.1) and (2.4), respectively. Suppose that $\sigma \in H^3(\Omega, \mathbb{S})$ and $u \in H^1(\Omega, \mathbb{R}^n)$. Then there holds that*

$$(3.24) \quad \|\sigma_h - \Pi_h \sigma\|_{H(\text{div})}^2 + \|u_h - P_h u\|_0^2 \leq Ch^3(\|\sigma\|_3^2 + \|u\|_1^2).$$

Proof. It is apparent that we can not derive the desired superconvergence directly from (3.9) and (3.16). We propose an iterative argument to show (3.24), which consists of the following steps:

Step 1: By (3.9) and (3.16), we can deduce the following initial superconvergence result:

$$(3.25) \quad \|\sigma_h - \Pi_h \sigma\|_0^2 + \|u_h - P_h u\|_0^2 \leq Ch^{2+\frac{1}{2}}(\|\sigma\|_3^2 + \|u\|_1^2).$$

Step 2: We show an intermediate superconvergence for $\|\text{div}(\sigma_h - \Pi_h \sigma)\|_0$ based on (3.1), which for convenience is recalled as follows

$$(3.26) \quad \begin{aligned} & \|\Pi_h \sigma - \sigma_h\|_{H(\text{div})} + \|P_h u - u_h\|_0 \\ & \leq C \sup_{0 \neq (\tau, v) \in \Sigma_h \times V_h} \frac{(A(\sigma - \Pi_h \sigma), \tau) + (u - P_h u, \text{div } \tau) - (\text{div}(\sigma - \Pi_h \sigma), v)}{\|\tau\|_{H(\text{div})} + \|v\|_0}. \end{aligned}$$

Since the second term on the right-hand side of (3.26) vanishes and the third term is already analyzed in (3.6), we only need to show a better bound for the first term. In fact, it follows from (2.1), (2.4) and (3.25) that

$$(3.27) \quad \begin{aligned} (A(\sigma - \Pi_h \sigma), \tau) &= (A(\sigma - \sigma_h), \tau) + (A(\sigma_h - \Pi_h \sigma), \tau) \\ &= (u_h - u, \text{div } \tau) + (A(\sigma_h - \Pi_h \sigma), \tau) \\ &= (u_h - P_h u, \text{div } \tau) + (A(\sigma_h - \Pi_h \sigma), \tau) \\ &\leq Ch^{1+\frac{1}{4}}(\|\sigma\|_3 + \|u\|_1)\|\tau\|_{\text{div}}. \end{aligned}$$

Consequently,

$$(3.28) \quad \|\Pi_h \sigma - \sigma_h\|_{H(\text{div})} \leq Ch^{1+\frac{1}{4}}(\|\sigma\|_3 + \|u\|_1).$$

We substitute this estimate into (3.19) to get an improved estimate as

$$(3.29) \quad \sum_{\ell=1}^n \sum_{i \neq k=1}^n \left(\frac{\partial \sigma_{kk}}{\partial x_i}, \frac{\partial(\sigma_{i\ell, h} - \Pi_{i\ell} \sigma_{i\ell})}{\partial x_\ell} \right) \leq Ch^{1+\frac{1}{4}}(\|\sigma\|_3^2 + \|u\|_1^2).$$

Step 3: We establish an improved estimate for the boundary term by putting the estimate $\|\sigma_h - \Pi_h \sigma\|_0$ of (3.25) into (3.21):

$$(3.30) \quad \sum_{F \in \mathcal{F}_{x_\ell} \cap \partial\Omega} \int_F \tau_{i\ell} \frac{\partial \sigma_{kk}}{\partial x_i} dF \leq Ch^{\frac{3}{4}}(\|\sigma\|_3^2 + \|u\|_1^2).$$

Step 4: We replace those corresponding estimates used in the proof of Lemma 3.8 by these improved estimates to obtain

$$(3.31) \quad (A(\sigma - \Pi_h \sigma), \sigma_h - \Pi_h \sigma_h) \leq Ch^{2+\frac{3}{4}}(\|\sigma\|_3^2 + \|u\|_1^2).$$

Thus we increase the order of convergence in (3.25) from $1 + 1/4$ to $1 + 1/4 + 1/8$. Now we go back to Step 1 and repeat this procedure to get another $1/16$ higher order of convergence. The iteration converges with (3.24). ■

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